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A Sharp Version of Henry's Theorem on Small Solutions

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A small solution of a linear autonomous retarded functional differential equation (rfde) is a solution that goes to zero faster than any exponential. Henry's theorem on small solutions states that there exists a time T—depending on the dimension and the delay of the equation—such that all small solutions vanish a.e. for $t \geqslant T$. In this paper we shall give an explicit characterisation for the smallest possible time T, in terms of properties of the specific kernel. This characterisation helps to establish new results concerning completeness and F-completeness of the generalized eigenfunctions of the infinitesimal generator of the C_0-semigroup associated with the linear autonomous rfde.

1. INTRODUCTION

This paper is a condensed version of the report [11] to which we refer for more information, additional results and detailed proofs.

Consider the following class of Volterra convolution integral equations:

$$x - \zeta * x = f, \tag{1.1}$$

where ζ is an $n \times n$-matrix valued function, and ζ and f are complex valued functions defined on $\mathbb{R}$ such that they vanish on $(-\infty, 0)$, are Lebesgue L^2-integrable on $[0, h]$ and are constant on $[h, \infty)$ and the convolution product $\zeta * x$ is defined by

$$\zeta * x(t) = \int_0^t \zeta(t-\theta)\, x(\theta)\, d\theta.$$

An application of the Banach fixed point theorem yields that the equation (1.1) has a unique solution $x \in L^2_{\text{loc}}(\mathbb{R}_+)$, the space of locally L^2-integrable functions on $[0, \infty)$. Note that the solution is continuous on $[h, \infty)$.

The class of Volterra convolution integral equations defined above is closely related to the class of linear retarded functional differential

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equations studied by Delfour and Manitius [3, 4, 9]. The following observations indicate that a proper treatment of the theory of linear autonomous rfde's starts with the study of the above class of Volterra convolution integral equations. First of all the equation (1.1) is well defined, whereas the linear autonomous rfde is a priori not well defined on the state space one likes to deal with. Second, it turns out that one can associate with the adjoint semigroup $\{T(t)^*\}$ of the C_0-semigroup $\{T(t)\}$ corresponding to a linear autonomous rfde a Volterra convolution integral equation of the above type. Finally, it turns out that equivalent results are much easier to prove for the Volterra convolution integral equation.

In this paper we shall prove a sharp version of Henry's theorem [8] on small solutions for the Volterra convolution integral equation (1.1). The corresponding result for the linear autonomous rfde and corollaries concerning completeness and F-completeness of the generalized eigenfunctions are only formulated; detailed proofs are given in [11].

2. A Sharp Version of Henry's Theorem

In this section we shall give a characterisation for the smallest possible time t_0 such that all small solutions vanish a.e. for $t \geqslant t_0$. This characterisation of t_0 is needed in order to establish the results concerning completeness and F-completeness of the generalized eigenfunctions stated in the next section.

Let $\Delta(z)$ denote the characteristic matrix function

$$\Delta(z) = zI - e^{-zh}\zeta(h) - z\int_0^h e^{-zt}\zeta(t)\,dt. \tag{2.1}$$

The matrix function $\Delta(z)$ appears in a natural way if one Laplace transforms the equation (1.1). Let $\det \Delta(z)$ denote the determinant of $\Delta(z)$. The function $\det \Delta(z)$ is an entire function of order 1 and because of the Paley–Wiener theorem of exponential type less than or equal to nh. Define ε by

$$\text{exponential type } \det \Delta(z) = nh - \varepsilon$$

Let $\operatorname{adj} \Delta(z)$ denote the matrix function of cofactors of $\Delta(z)$. Since the cofactors $C_{ij}(z)$ are $(n-1)\times(n-1)$ subdeterminants of $\Delta(z)$, the exponential type of the cofactors is less than or equal to $(n-1)h$. Define σ by

$$\max_{1 \leqslant i,j \leqslant n} \text{exponential type } C_{ij}(z) = (n-1)h - \sigma.$$

DEFINITION 2.1. A small solution x of (1.1) is a (non almost everywhere zero) solution x such that

$$\lim_{t \to \infty} e^{kt} x(t) = 0,$$

for all $k \in \mathbb{R}$.

We can now state and prove our main result.

THEOREM 2.2. *All small solutions of* (1.1) *vanish almost everywhere for* $t \geqslant \varepsilon - \sigma$ *and* $\varepsilon - \sigma$ *is the smallest possible time with this property.*

As an application we have:

THEOREM 2.3. *There are no small solutions if and only if exponential type* $\det \Delta(z)$ *is equal to nh.*

Proof of Theorem 2.2. Let x be a small solution then $\int_0^\infty e^{-zt} x(t)\, dt$ is an entire function and by the Plancherel theorem L^2-integrable along the imaginary axis. Laplace transformation of the equation (1.1) yields

$$\Delta(z) \int_0^\infty e^{-zt} x(t)\, dt = z \int_0^h e^{-zt} f(t)\, dt + e^{-zh} f(h).$$

Hence

$$\det \Delta(z) \int_0^\infty e^{-zt} x(t)\, dt = \operatorname{adj} \Delta(z) \left\{ z \int_0^h e^{-zt} f(t)\, dt + e^{-zh} f(h) \right\}. \quad (2.2)$$

Since the quotient of two functions of exponential type is again of exponential type provided it is entire, $\int_0^\infty e^{-zt} x(t)\, dt$ is of exponential type. We now need a lemma.

LEMMA 2.4. *Let F and G be entire functions of exponential type such that F and G are* $O(z^m)$, $m \in \mathbb{Z}$, *in the closed right half plane. Then*

exponential type $(F.G) =$ *exponential type* $(F) +$ *exponential type* (G).

Proof. An applicaton of the Ahlfors–Heins theorem (7.2.6) of Boas [2]. ▮

By Lemma 2.4 the right-hand side of (2.2) has exponential type less than or equal to $nh - \sigma$. And so again by Lemma 2.4 $\int_0^\infty e^{-zt} x(t)$ has finite exponential type η and $\eta \leqslant nh - \sigma - (nh - \varepsilon) = \varepsilon - \sigma$. Hence, by the Paley–Wiener theorem

$$\int_0^\infty e^{-zt} x(t)\, dt = \int_0^\eta e^{-zt} x(t)\, dt,$$

and $x(t) = 0$ a.e. for all $t \geqslant \varepsilon - \sigma$.

In the following we shall call functions of the form $\int_0^\rho e^{-zt}\psi(t)\,dt$, $\rho \in \mathbb{R}_+$ and $\psi \in L^2[0, \rho]$, Paley–Wiener functions. To prove the fact that $\varepsilon - \sigma$ is the smallest possible time with this property we shall contruct a small solution x such that $x \not\equiv 0$ a.e. in any neighbourhood of $\varepsilon - \sigma$. Laplace transformation yields that it suffices to construct a Paley–Wiener function F of exponential type $\varepsilon - \sigma$ such that

$$\Delta(z)\, F(z) = c + q(z),$$

where $c \in \mathbb{C}^n$ and q is a Paley–Wiener function of exponential type $\leqslant h$.

Choose a column of the matrix function adj $\Delta(z)$ such that one of the elements of this column is the cofactor of maximal exponential type $(n-1)\,h - \sigma$. Since the arguments given below can be repeated for all other columns we may assume that we can choose the first column

$$\begin{pmatrix} C_{11}(z) \\ \vdots \\ C_{n1}(z) \end{pmatrix}$$

of adj $\Delta(z)$. Then

$$\Delta(z) \begin{pmatrix} C_{11}(z) \\ \vdots \\ C_{n1}(z) \end{pmatrix} = \begin{pmatrix} \det \Delta(z) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.3}$$

We have to consider two cases:

I. $\varepsilon \leqslant (n-1)\,h$;

II. $(n-1)\,h < \varepsilon \leqslant nh$.

Case I. Suppose $\varepsilon \leqslant (n-1)\,h$. Let for $1 \leqslant j \leqslant n$, c_j denote the Taylor expansion of C_{j1} of order $n-1$ in 0. Then the functions F_j defined by

$$F_j(z) = \frac{C_{j1}(z) - c_j(z)}{z^n},$$

$1 \leqslant j \leqslant n$, are entire. Let

$$\Delta(z) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}.$$

The functions d_j, $1 \leqslant j \leqslant n$, are polynomials of degree n with coefficients being constants plus Paley–Wiener functions of exponential type $\leqslant h$.

Furthermore

$$\Delta(z)\begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix} = \frac{1}{z^n}\begin{pmatrix} \det \Delta(z) - d_1 \\ -d_2 \\ \vdots \\ -d_n \end{pmatrix}. \tag{2.4}$$

Since $\det \Delta(z)$ is a polynomial of degree n with coefficients being constants plus Paley–Wiener functions we have by the Paley–Wiener theorem that the right-hand side of (2.4) can be written as follows:

$$c + \int_0^{nh-\varepsilon} e^{-zt} h(t)\, dt,$$

where $c \in \mathbb{C}^n$ and $h \in L^2([0, nh-\varepsilon]; \mathbb{C}^n)$. Furthermore, the cofactors are polynomials of degree $(n-1)$ with coefficients being constants plus Paley–Wiener functions. Hence, F is a Paley–Wiener function and by the Paley–Wiener theorem we have

$$F(z) = \int_0^{(n-1)h-\sigma} e^{-zt} \psi(t)\, dt,$$

where $\psi \in L^2([0, (n-1)h-\sigma]; \mathbb{C}^n)$. Therefore the equation (2.4) can be rewritten as follows:

$$\Delta(z) \int_0^{(n-1)h-\sigma} e^{-zt} \psi(t)\, dt = c + \int_0^{nh-\varepsilon} e^{-zt}\, h(t)\, dt. \tag{2.5}$$

Hence, the function ψ satisfies the equation

$$x - \zeta * x = q, \tag{2.6}$$

where $q(t) = c + \int_0^t h(s)\, ds$ for $0 \leqslant t \leqslant nh - \varepsilon$ and constant on $[nh-\varepsilon, \infty)$.

Since the solution of (2.6) can be written as $q - R * q$, where R is defined as the solution of the equation $R = R * \zeta - \zeta$, we obtain

$$\frac{d\psi}{dt} \in L^2[0, (n-1)h - \sigma].$$

Rewrite the equation (2.5) as follows

$$e^{-((n-1)h-\varepsilon)z} \Delta(z) \int_0^{\varepsilon-\sigma} e^{-zt} \psi((n-1)h - \varepsilon + t)\, dt$$

$$= c + \int_0^{nh-\varepsilon} e^{-zt} h(t)\, dt - \Delta(z) \int_0^{(n-1)h-\varepsilon} e^{-zt} \psi(t)\, dt. \tag{2.7}$$

Since the right-hand side of (2.7) has exponential type less than or equal to $nh-\varepsilon$ we have by Lemma 2.4 that

$$\Delta(z)\int_0^{\varepsilon-\sigma} e^{-zt}\psi((n-1)h-\varepsilon+t)\,dt \tag{2.8}$$

has exponential type less than or equal to h. Furthermore, since $d\psi/dt\in L^2[0,(n-1)h-\sigma]$ partial integration yields that (2.8) can be written as a constant plus a Paley–Wiener function. Hence

$$\Delta(z)\int_0^{\varepsilon-\sigma} e^{-zt}\psi((n-1)h-\varepsilon+t)\,dt=b+\int_0^h e^{-zt}\iota(t)\,dt,$$

where $b\in\mathbb{C}^n$ and $\iota\in L^2[0,h]$. Hence $\psi((n-1)h-\varepsilon+\cdot)$ is a small solution such that

$$\psi((n-1)h-\varepsilon+\cdot)\not\equiv 0 \text{ a.e. in any neighborhood of } \varepsilon-\sigma.$$

This yields $\alpha=\varepsilon-\sigma$.

Case II. Suppose $(n-1)h<\varepsilon\leqslant nh$. In this case $\tau=$ exponential type $\det\Delta(z)<h$. Multiply both sides of the equation (2.3) by

$$\int_0^{h-\tau} e^{-zt}\,dt$$

to obtain

$$\Delta(z)\begin{pmatrix}\tilde{C}_{11}\\ \vdots\\ \tilde{C}_{n1}\end{pmatrix}=\begin{pmatrix}G(z)\\ 0\\ \vdots\\ 0\end{pmatrix},$$

where $G(z)=\int_0^{h-\tau} e^{-zt}\,dt\det\Delta(z)$ has type h, $\tilde{C}_{j1}=\int_0^{h-\tau} e^{-zt}\,dt\,C_{j1}$, $1\leqslant j\leqslant n$, and the function $\tilde{C}$ has type $\varepsilon-\sigma$. The same arguments as used in Case I applied to the function $\tilde{C}$ yields

$$\Delta(z)\int_0^{\varepsilon-\sigma} e^{-zt}\tilde{\psi}(t)\,dt=\tilde{b}+\int_0^h e^{-zt}\tilde{h}(t)\,dt.$$

Hence, $\tilde{\psi}$ is a small solution such that

$$\tilde{\psi}\not\equiv 0 \qquad \text{a.e.}$$

in any neighborhood of $\varepsilon-\sigma$. This yields $\alpha=\varepsilon-\sigma$. ∎

Proof of Theorem 2.3. Because of Theorem 2.2 it suffices to prove

$$\forall \varepsilon > 0 \colon \sigma < \varepsilon.$$

Suppose $\sigma = \varepsilon$. We shall calculate the exponential type of $\det \operatorname{adj} \Delta(z)$ in two different ways. Since $\sigma = \varepsilon$ we have

$$\text{exponential type } \det \operatorname{adj} \Delta(z) \leqslant n((n-1h-\varepsilon) = (n-1)(nh-\varepsilon) - \varepsilon.$$

On the other hand by Lemma 2.4 we have

$$\begin{aligned}\text{exponential type } \det \operatorname{adj} \Delta(z) &= \text{exponential type } (\det \Delta(z))^{(n-1)}\\ &= (n-1)(nh-\varepsilon).\end{aligned}$$

Hence

$$(n-1)(nh-\varepsilon) \leqslant (n-1)(nh-\varepsilon) - \varepsilon,$$

which is a contradiction if $\varepsilon > 0$. ∎

Remark 2.5. Using the notation introduced above, Henry's theorem on small solutions for the Volterra convolution integral equation (1.1) can be stated as follows: All small solutions of (1.1) vanish a.e. for $t \geqslant \varepsilon$. A slight modification of Henry's proof yields: All small solutions of (1.1) vanish a.e. for $t \geqslant \varepsilon - \sigma$. The hard part in proving Theorem 2.2 is to construct a small solution which does not vanish a.e. in any neighbourhood of $\varepsilon - \sigma$. This property of being minimum of $\varepsilon - \sigma$ yields the improvement of Henry's theorem.

3. The Linear Autonomous rfde

Consider the following linear autonomous rfde,

$$\begin{aligned}\frac{dx}{dt}(t) &= Lx_t, \qquad t \geqslant 0,\\ x(0) &= \phi^0,\\ x_0 &= \phi^1,\end{aligned} \tag{3.1}$$

where x_t is the translation of x over t considered as a function on $[-h, 0]$, L is a continuous mapping from $C[-h, 0]$ into $\mathbb{C}^n$ and $(\phi^0, \phi^1) \in M_2 = \mathbb{C}^n \times L^2[-h, 0]$. Because of the Riesz representation theorem there is a matrix function ζ of bounded variation on $[0, h]$ (suitably normalized) such that

$$L\phi = \int_0^h \phi(-t)\, d\zeta(t), \tag{3.2}$$

for all $\phi \in C[-h, 0]$. Translation along the solution defines a C_0-semigroup $\{T(t)\}$ on M_2 by

$$T(t)\,\phi = (x(t; \phi), x_t(\cdot\,; \phi)). \tag{3.3}$$

Let α denote the ascent of $\{T(t)\}$, i.e., $\alpha = \inf\{t \mid \text{for all } \varepsilon > 0\ N(T(t)) = N(T(t+\varepsilon))\}$. In [11] we proved, motivated by earlier results of Delfour and Manitius [4, 9], the existence of a bounded invertible operator Ω such that $\Omega T(t)\,\Omega^{-1} = S(t)$, where $\{S(t)\}$ is the C_0-semigroup associated to the Volterra convolution integral equation (1.1) defined by

$$x_t - \zeta * x_t = S(t)\,f, \tag{3.4}$$

where $x_t(\cdot) = x(t + \cdot)$ on $[0, \infty)$. Using this result we obtain by Theorem 2.2

THEOREM 3.1. *The ascent α of $\{T(t)\}$ is equal to $\varepsilon - \sigma$.*

Moreover, since $\varepsilon(\zeta) = \varepsilon(\zeta^*)$ and $\sigma(\zeta) = \sigma(\zeta^*)$ we obtain

COROLLARY 3.2. *The ascent α of $\{T(t)\}$ is equal to the ascent δ of $\{T(t)^*\}$.*

Note that this corollary generalizes a result of Bartosiewicz [1], who proved for a special class of ζ that $\alpha = 0$ if and only if $\delta = 0$.

An application of a result of Manitius [9] that completeness holds if and only if $\delta = 0$ yields

COROLLARY 3.3. *Completeness holds if and only if exponential type of* $\det \Delta(z)$ *is equal to nh. Or, equivalently, completeness holds if and only if there are no small solutions.*

Delfour and Manitius introduced in their papers [4, 9] also the concept of F-completeness. The following corollary yields an easy to verify necessary and sufficient condition for F-completeness.

COROLLARY 3.4. *F-completeness holds if and only if $\varepsilon - \sigma \leqslant h$.*

Or, equivalently, F-completeness holds if and only if all small solutions vanish almost everywhere on $[0, \infty)$ (are in the kernel of F; see [11]).

Remark 3.5. All the results given in this section also hold for the more general class of rfde considered by Delfour in [3].

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